JOURNAL OF APPROXIMATION THEORY 12, 332–351 (1974)

Simultaneous Approximation of Compact Sets by Elements of Convex Sets in Normed Linear Spaces

KIM-PIN LIM

Department of Mathematics, University of Malaya, Kuala Lumpur, Malaysia

Communicated by E. W. Cheney

1. INTRODUCTION

Starting from a problem concerning the uniform approximation of a function and its derivatives, Moursund [10, 11] has recently considered the problem of finding, for a real continuous function $f(x)$ in $C^r[a, b]$, an algebraic polynomial $g_0(x) = \sum_{i=1}^{n} a_i^0 x^{i-1}$ of degree $n - 1$ which minimizes

$$\max_{0 \leqslant i \leqslant r} \left(\max_{x \in [a,b]} | D^i(f(x) - g(x))| \right).$$

The nature and number of the extreme points of best approximation were investigated. In the special case of approximation of a function and its first derivative by algebraic polynomials a uniqueness theorem was obtained. More recently, Dunham [5] studied the problem of simultaneously approximating elements of a given set F by elements of a family of real-valued functions, unisolvent of degree N, on a compact interval of the real line. He considered the cases (i) F consists of one bounded real-valued function, (ii) F consists of an upper semicontinuous real-valued function f^+ and a lower semicontinuous real-valued function f^-, with $f^+ \geqslant f^-$ pointwise, and (iii) F consists of a finite number of continuous real-valued functions. Later on, Diaz and Mclaughlin [3] showed that analogous results hold for any nonempty family F; in fact, they pointed out that the general problem of simultaneous approximation of a family F by means of functions from a family G is equivalent to the problem of the simultaneous approximation of certain functions F^- and F^+ with $F^- \leqslant F^+$, where

$$F^+(x) = \inf_{\delta > 0} \sup_{0 \leqslant |x-y| < \delta} \sup_{f \in F} f(y)$$

and

$$F^-(x) = \sup_{\delta > 0} \inf_{0 \leqslant |x-y| < \delta} \inf_{f \in F} f(y)$$

for $a \leqslant x \leqslant b$.

At this stage there arose, in a natural way, the question: Could all these results be particular cases of a more general theory? The answer is affirmative. Namely, the above sets G and F can be considered as subsets of a more general "space", and the derivatives of their elements can be considered as continuous operators on general spaces. In fact, we shall deal with a still more general problem: Let X, Y be two real normed linear spaces, F a compact subset of X, G a convex subset of X and $A : X \to Y$ a continuous operator. The question is to find a $g' \in G$ which minimizes

$$\max(\max_{f \in F} p_1(f - g), \max_{f \in F} p_2(Af - Ag)),$$

where $p_1(\cdot)$ and $p_2(\cdot)$ are given continuous seminorms on X and Y, respectively. The theory which we will develop for this problem is close to those in the theory of uniform approximation of a function and its derivatives. The sufficiency of a Kolmogorov-type condition for a best approximation for operators A subject to no restrictions and the necessity of that condition for A subject to "closed-sign" property are given. In case G is a finite-dimensional subspace, we obtain a similar result to that in [12, p. 170]. The application to various kind of spaces will be discussed elsewhere.

A summary of notation is given as follows: Let X^*, Y^* be the dual spaces of X and Y, respectively. The value of the continuous linear functional k in X^* (or Y^*) at x in X (or Y) will be denoted as (k, x). Let $K(\tilde{K})$ be a subset of X^* (Y^*, respectively) which is symmetrical, $\sigma(X^*, X)$-compact ($\sigma(Y^*, Y)$-compact) and norm-bounded. We define continuous seminorms p_1, p_2 on X, Y, respectively as

$$p_1(f) = \max_{k \in K} (k, f) \qquad \text{for} \quad f \text{ in } X,$$

$$p_2(y) = \max_{\tilde{k} \in \tilde{K}} (\tilde{k}, y) \qquad \text{for} \quad y \text{ in } Y.$$

Let F be a compact subset of X (in norm topology). We introduce, for $g \in X$,

$$d_1(g) = \max_{f \in F} p_1(f - g),$$

$$d_2(g) = \max_{f \in F} p_2(Af - Ag),$$

and

$$d_F(g) = \max(d_1(g), d_2(g)).$$

Let G be a closed convex subset of X. We seek an element g' in G such that

$$d_F(g') = \inf_{g \in G} d_F(g).$$

Such an element will be called a simultaneous best approximation to F from G, or more briefly, a "best approximation."

To answer the question of the existence of g', we first note that the functions $d_1(g)$ and $d_2(g)$ are the suprema of families of continuous functions, therefore, lower semicontinuous [7, p. 89]. It follows that $d_F(g) = \max(d_1(g), d_2(g))$ is again lower semicontinuous. This ensures that $d_F(g)$ attains its infimum on a compact set. Thus, we have the following:

LEMMA 1.1. *Let G be an n-dimensional subspace of X. Assume that the restriction of $p_1(\cdot)$ to G is a norm. Then there exists $g' \in G$ such that*

$$d_F(g') = \inf_{g \in G} d_F(g).$$

Proof. Let g_i be a sequence in G such that $\lim d_F(g_i) = \inf_{g \in G} d_F(g)$. Moreover,

$$p_1(g_i) \leqslant \max_{f \in F} p_1(f - g_i) + \max_{f \in F} p_1(f) \leqslant M,$$

for some real M. Since $\max_{f \in F} p_1(f - g_i) \leqslant d_F(g_i)$ and $\max_{f \in F} p_1(f)$ is fixed. As the restriction of $p_1(\cdot)$ is a norm, so $\{g_i\}$ is a bounded sequence in G. Hence there exists $g' \in G$ such that g_i converges to g'. Furthermore for each i,

$$0 \leqslant d_F(g') - \inf_{g \in G} d_F(g) \leqslant d_F(g') - d_F(g_i) + (d_F(g_i) - \inf_{g \in G} d_F(g))$$

since the term in bracket tends to zero as $i \to \infty$ and also, by semicontinuity of $d_F(\cdot)$, $d_F(g') - d_F(g_i) \to 0$ as $i \to \infty$, this shows that

$$d_F(g') = \inf_{g \in G} d_F(g),$$

which proves the theorem.

Obviously, in the special case when g' minimizes $d_1(g)$ or $d_2(g)$ and $d_F(g') = d_1(g')$ (or $d_2(g')$), then the problem is solved, with g' a best approximation, since in any case

$$\max(\inf_{g \in G} d_1(g), \inf_{g \in G} d_2(g)) \leqslant \inf_{g \in G} \max(d_1(g), d_2(g)).$$

Moreover, in this case equality holds here. In the more general case where inequality holds, we have the following useful result.

LEMMA 1.2. *Assume that*

$$\max(\inf_{g \in G} d_1(g), \inf_{g \in G} d_2(g)) < \inf_{g \in G} \max(d_1(g), d_2(g)).$$

If $d_2(g) = \max_{f\in F} p_2(Af - Ag)$ is convex for all g in G and g′ is a best approximation, then $d_1(g') = d_2(g')$.

Proof. Assume that $d_2(g') - d_1(g') = \epsilon > 0$ (for the case $\epsilon < 0$, the proof follows in the same way). Then g' cannot minimize d_2, for this would lead to a contradiction of the hypothesis. Define a set

$$U = \{g \in G : p_1(g - g') \leqslant \tfrac{1}{2}\epsilon\}.$$

Then

$$d_1(g) \leqslant \max_{f\in F} \{p_1(f - g') + p_1(g' - g)\} \leqslant d_1(g') + \tfrac{1}{2}\epsilon \qquad \text{for} \quad g \in U.$$

Moreover, since $d_2(g)$ is convex and g' does not minimize $d_2(g)$, by the global minimum property of convex functions [2, p. 25], there exists $g_1 \in U$ such that

$$d_2(g_1) < d_2(g').$$

This shows that g_1 is a better approximation than g', which contradicts the hypothesis. The proof is thus completed.

Since the convexity assumption on $d_2(g)$ holds, for example, when A is a linear operator, it is clear that in a large class of problems having best approximations we have $d_1(g') = d_2(g')$. That this is not always the case however is illustrated by the following example:

EXAMPLE 1. Let $X = Y = C[-1, 1]$ with Chebyshev norm. Obviously Chebyshev norm is generated by the set $\{\pm$(point evaluated functionals at $x) : -1 \leqslant x \leqslant 1\}$ of C^*. Now, define an operator A as follows: for any $h(x)$ in $C[-1, 1]$,

$$A(h)(x) = \begin{cases} 1 & h(x) = 1/2; \\ 3/4 & h(x) = 3/4; \\ 3/4 & h(x) = 1/4; \\ (7/4) - 4h(x) & h(x) \leqslant 1/4; \\ h(x) & h(x) \geqslant 3/4; \\ (3/2) - h(x) & 1/2 \leqslant h(x) \leqslant 3/4; \\ (1/2) + h(x) & 1/4 \leqslant h(x) \leqslant 1/2. \end{cases}$$

It is easy to check that A is a continuous mapping from $C[-1, 1]$ into itself. Suppose $F = \{f(x) = 1 - x^2\}$ is to be approximated by real constants. Then,

$$d_F(a) = \max(\| 1 - x^2 - a \|_\infty , \| A(1 - x^2) - Aa \|_\infty).$$

As is evident from the following figure $a = \frac{1}{2}$ is a best approximation, and $d_F(\frac{1}{2}) = \frac{3}{4} = d_2(\frac{1}{2}) \neq d_1(\frac{1}{2}) = \frac{1}{2}$. Moreover, we have

$$\max(\inf_{a\in R} d_1(a), \inf_{a\in R} d_2(a)) < \inf_{a\in R} d_F(a).$$

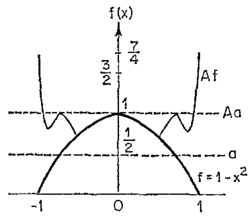

FIGURE 1

Finally, we consider a particular situation which leads trivially to a best approximation as defined above, and which we wish to exclude. Suppose that, for some $g_0 \in G$ and $k \in K$ (or $\tilde{k} \in \tilde{K}$), there exists $f_1, f_2, \in F$ such that

$$(k, f_1 - g_0) = d_F(g_0) \qquad \text{and} \quad (k, f_2 - g_0) = -d_F(g_0)$$

(or $(\tilde{k}, Af_1 - Ag_0) = d_F(g_0)$ and $(\tilde{k}, Af_2 - Ag_0) = -d_F(g_0)$). Then g_0 is a best approximation, as no approximation can make the error smaller at k (or $\tilde{k}$). For example, suppose $X = C^{(\gamma)}[0, \pi](\gamma \geqslant 1)$ endowed with norm $\max(\|f\|_\infty, \|Df\|_\infty)$ and $Y = C[0, \pi]$, $F = \{e^{-x}, \sin x - 1\}$ and A is the first derivative operator. Let $p_1(f) = p_2(f) = \max_{x\in[0,\pi]} |f(x)|$. We consider that F is to be approximated by $ax + b$ where a, b are real numbers. It is obvious that when $a = 0$, $b = 0$, there exist $x_0 = x_1 = 0$ such that

$$d_F(0) = \max_{f\in F} |f(0)| = \max_{f\in F} |Df(0)| = 1$$

and

$$(e^{-0})(\sin 0 - 1) < 0$$
$$(-e^{-0})(\cos 0) < 0.$$

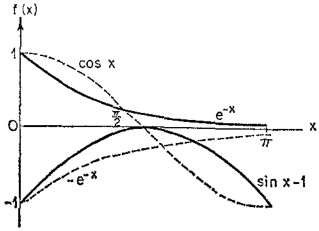

FIGURE 2

Thus, $a = 0$, $b = 0$ is a best approximation. As one can easily check from the above figure, there is no other a, b that can make both errors smaller at the point $x_0 = x_1 = 0$.

This situation has been discussed where a point such as g_0 is called a "straddle phenomenon." In this chapter, unless otherwise stated, the term "best approximation" will exclude stranddle phenomena.

2. General Characterization Theorem

Given $g' \in G$ define the following subsets of K and $\tilde{K}$, respectively, as

$$B_{g'} = \{k \text{ in } K : \exists f \text{ in } F,\ (k, f - g') = d_F(g')\};$$
$$\tilde{B}_{g'} = \{\tilde{k} \text{ in } \tilde{K} : \exists f \text{ in } F,\ (\tilde{k}, Af - Ag') = d_F(g')\}.$$

Since

$$d_1(g) = \max_{f \in F} \max_{k \in K} (k, f - g) = \max_{k \in K} \max_{f \in F} (k, f - g),$$

so by compactness there exist $k \in K, f \in F$ such that $d_1(g) = (k, f - g)$. Similarly, for $d_2(g)$.

Also, if $d_1(g') \neq d_2(g')$, one of $B_{g'}$, $\tilde{B}_{g'}$ is empty.

Lemma 2.1. *Let g' in G and closed subsets $M \subset K$, $\tilde{M} \subset \tilde{K}$ be such that*

$$\max_{f \in F} (k, f - g') \geqslant 0 \qquad \textit{for all} \quad k \textit{ in } M,$$

$$\max_{f \in F} (\tilde{k}, Af - Ag') \geqslant 0 \qquad \textit{for all} \quad \tilde{k} \textit{ in } \tilde{M},$$

and

$$\inf_{k \in M, \tilde{k} \in \tilde{M}} ((k, g - g'), (\tilde{k}, Ag - Ag')) \leqslant 0 \qquad \textit{for all} \quad g \textit{ in } G.$$

Then

$$\eta = \inf_{g \in G} d_F(g) \geqslant \inf_{k \in M, \tilde{k} \in \tilde{M}} (\max_{f \in F} (k, f - g'), \max_{f \in F} (\tilde{k}, Af - Ag')).$$

Proof. In case $\max_{f \in F}(k, f - g') = 0$ for some k in M or

$$\max(\tilde{k}, Af - Ag') = 0$$

for some $\tilde{k}$ in $\tilde{M}$, the lemma is trivial. We assume that $\max_{f\in F}(k, f - g') > 0$ for all k in M and $\max_{f\in F}(k, Af - Ag') > 0$ for all $\tilde{k}$ in M. Suppose that

$$\inf_{k\in M,\tilde{k}\in\tilde{M}} (\max_{f\in F}(k, f - g'), \max_{f\in F}(\tilde{k}, Af - Ag')) > \eta.$$

Then, there exists g in G such that

$$\eta \leqslant d_F(g) < \inf_{k\in M,\tilde{k}\in\tilde{M}} (\max_{f\in F}(k, f - g'), \max_{f\in F}(\tilde{k}, Af - Ag')).$$

This implies that

$$d_1(g) < \max_{f\in F}(k, f - g') \qquad \text{for all} \quad k \text{ in } M$$

and

$$d_2(g) < \max_{f\in F}(\tilde{k}, Af - Ag') \qquad \text{for all} \quad \tilde{k} \text{ in } \tilde{M}.$$

Therefore, for k in M, there exists f_1 (depending on k) in F such that

$$(k, f_1 - g) < (k, f_1 - g'),$$

i.e.,

$$(k, g - g') > 0,$$

and, for each $\tilde{k}$ in $\tilde{M}$, there exists f_2 (depending on $\tilde{k}$) in F such that

$$(\tilde{k}, Af_2 - Ag) < (\tilde{k}, Af_2 - Ag'),$$

i.e.,

$$(\tilde{k}, Ag - Ag') > 0.$$

Thus we have

$$(k, g - g') > 0 \qquad \text{for all} \quad k \in M$$

and

$$(\tilde{k}, Ag - Ag') > 0 \qquad \text{for all} \quad \tilde{k} \in \tilde{M}.$$

This contradicts the hypothesis, which proves the lemma.

THEOREM 2.1. *If $g' \in G$ is such that $d_1(g') = d_2(g')$ and*

$$\inf_{k\in B_{g'},\tilde{k}\in\tilde{B}_{g'}} ((k, g - g'), (\tilde{k}, Ag - Ag')) \leqslant 0 \qquad \textit{for} \quad g \in G, \tag{2.1}$$

then g' is a best approximation.

In fact, this is an immediate consequence of the Lemma 2.1.

We know that if $d_1(g') \neq d_2(g')$, then one or other of the sets $B_{g'}$ or $\tilde{B}_{g'}$ is empty, and in all of the following theorems of this chapter we shall accordingly, as in Theorem 2.1, explicitly assume $d_1(g') = d_2(g')$.

EXAMPLE 2. Suppose $X = Y = C[0, 1]$ and define $A : X \to Y$ by $Ah(x) = xh^2(x)$. Furthermore, assume that $K = \tilde{K} = \{\pm$ point evaluation functionals at $x : 0 \leqslant x \leqslant 1\}$, then

$$p_1(f) = p_2(f) = \|f\|_\infty = \max_{x \in [0,1]} |f(x)|.$$

Suppose F consists only of the function $f(x) = x$ and is to be approximated by real constants. Hence

$$d_F(a) = \max(\| x - a \|_\infty , \| x^3 - a^2 x \|_\infty).$$

It is easy to check that the best approximation for $d_F(a)$ is

$$a = \frac{-1 + \sqrt{5}}{2}$$

and

$$d_F\left(\frac{-1 + \sqrt{5}}{2}\right) = \frac{-1 + \sqrt{5}}{2} = d_1\left(\frac{-1 + \sqrt{5}}{2}\right) = d_2\left(\frac{-1 + \sqrt{5}}{2}\right).$$

Moreover, the set B_a of extremal functionals consists only of a negative point evaluation functional at $x = 0$, i.e., $B_a = \{\hat{x} : (\hat{x}, h) = -h(x)$, for all $h \in C[0, 1]$ and $x = 0\}$. Similarly, $\tilde{B}_a$ consists only of a positive point evaluation functional at $x = 1$, i.e., $\tilde{B}_a = \{\hat{x} : (\hat{x}, h) = h(x)$ for all $h \in C[0, 1]$ and $x = 1\}$. However, for some $c \in R$ such as $c = -2$ we have

$$(\hat{x}, c - a) = -\left(-2 - \left(\frac{-1 + \sqrt{5}}{2}\right)\right) > 0 \qquad \text{for} \quad \hat{x} \in B_a ,$$

and

$$(\hat{x}, Ac - Aa) = 4 - \left(\frac{-1 + \sqrt{5}}{2}\right)^2 > 0 \qquad \text{for} \quad \hat{x} \in \tilde{B}_a .$$

Hence, the inequality (2.1) is not fulfilled.

In view of this negative result, it appears that the necessity of condition (2.1) can be established only under further restrictive hypotheses. We now define a property which, as we shall prove, implies that the condition (2.1) is also necessary. This property is a generalization of the closed-sign property introduced by Dunham [6] in 1969.

DEFINITION 2.1. Let G be a convex subset of X. The continuous map A from X into Y is said to have the closed-sign property at $g \in G$, if, for any $h \in G$, and closed subset $W \subset \tilde{K}$ such that $(w, Ah - Ag) \neq 0$ for $w \in W$, there exists a $1 > \delta > 0$ such that, for $t \in (0, \delta]$

$$\operatorname{sgn}(w, Ah - Ag) = \operatorname{sgn}(w, Ag_t - Ag) \text{ for all } w \in W,$$

where $g_t = g + t(h - g)$.

We shall say that A has the closed-sign property on G if A has it at all $g \in G$.

To illustrate the definition, let $X = Y = C[0, 1]$ and define $A : X \to Y$ by $Ah = h^2$, convex subset $G = \{ax : \text{for real } a \geqslant 0\}$ and

$$\tilde{K} = \{\pm \hat{x} : x \in [0, 1]\},$$

where $\hat{x}$ is a point evaluation functional at x. Then, for any a_1x, a_2x in G and closed subset W of $\tilde{K}$, such that

$$(\hat{x}, Aa_1x - Aa_2x) = \pm (a_1 - a_2)(a_1 + a_2)\, x^2 \neq 0 \text{ for } \hat{x} \in W.$$

We have

$$(\hat{x}, Aa_tx - Aa_2x) = \pm\, t(a_1 - a_2)(2a_2 + t(a_1 - a_2))\, x^2,$$

where $a_t = a_2 + t(a_1 - a_2)$.

For the case $a_2 = 0$, it is clear $\operatorname{sgn}(\hat{x}, Aa_1x - Aa_2x) = \operatorname{sgn}(\hat{x}, Aa_tx - Aa_2x)$ for $t > 0$. Otherwise, there exists a $1 > \delta > 0$ such that $2a_2 + t(a_1 - a_2) > 0$ for all $t \in (0, \delta]$. Hence, for such δ, we have $\operatorname{sgn}(\hat{x}, Aa_1x - Aa_2x) = \operatorname{sgn}(\hat{x}, Aa_tx - Aa_2x)$ for all $\hat{x} \in W$, $t \in (0, \delta]$. This shows that A has the closed-sign property on G.

Let us return to the Example 2. It is easy to check that A fails to have the closed-sign property at $a = (-1 + \sqrt{5})/2$. Further, we have already shown that the condition (2.1) in Theorem 2.1 is not a necessary condition for g' to be a best approximation. In fact these two statements are related, as is shown by the following theorem:

THEOREM 2.2. *Suppose that A has the closed-sign property at $g' \in G$ and $d_1(g') = d_2(g')$. Then if g' is a best approximation,*

$$\inf_{k \in B_{g'}, \tilde{k} \in \tilde{B}_{g'}} ((k, g - g'), (\tilde{k}, Ag - Ag')) \leqslant 0 \qquad \textit{for} \quad g \in G.$$

Proof. Suppose there exists $g_1 \in G$ such that

$$\inf_{k \in B_{g'}, \tilde{k} \in \tilde{B}_{g'}} ((k, g_1 - g'), (\tilde{k}, Ag_1 - Ag')) > 0. \tag{2.2}$$

Then we will show that g' is not a best approximation. Since $B_{g'}, \tilde{B}_{g'}$ is $\sigma(X^*, X)$-compact ($\sigma(Y^*, Y)$-compact, respectively), from inequality (2.2), we may conclude that there are relatively open subsets U, $\tilde{U}$ of K, $\tilde{K}$ containing $B_{g'}$, $\tilde{B}_{g'}$ such that

$$\inf_{k\in U,\tilde{k}\in\tilde{U}} ((k, g_1 - g'), (\tilde{k}, Ag_1 - Ag')) \geqslant a \qquad \text{for some} \quad a > 0.$$

Since U, $\tilde{U}$ are relatively open in K, $\tilde{K}$ and $B_{g'} \subset U$, $\tilde{B}_{g'} \subset \tilde{U}$, there exists a real $c > 0$ such that

$$d_1(g') - \max_{k\in K\setminus U} \max_{f\in F} (k, f - g') \geqslant c, \tag{2.3}$$

$$d_2(g') - \max_{\tilde{k}\in \tilde{K}\setminus \tilde{U}} \max_{f\in F} (\tilde{k}, Af - Ag') \geqslant c. \tag{2.4}$$

Hence, for $k \in U$, $0 < t < 1$, we have

$$\begin{aligned}\max_{f\in F} (k, f - g_t) &= \max_{f\in F} (k, f - g') - t(k, g_1 - g') \\ &< \max_{f\in F} (k, f - g') \leqslant d_1(g'),\end{aligned} \tag{2.5}$$

where $g_t = g' + t(g_1 - g')$.

For $k \in K\setminus U$,

$$\begin{aligned}\max_{f\in F} (k, f - g_t) &\leqslant \max_{f\in F} (k, f - g') + tp_1(g_1 - g') \\ &\leqslant d_1(g') - c + tp_1(g_1 - g') < d_1(g')\end{aligned} \tag{2.6}$$

for $0 < t$ sufficiently small.

Let $\bar{\tilde{U}}$ be the closure of $\tilde{U}$. Since A is a continuous map, we have

$$(\tilde{k}, Ag_1 - Ag') \geqslant a \, \forall \tilde{k} \in \bar{\tilde{U}}.$$

Furthermore A has the closed-sign property at g', so for $0 < t$ sufficiently small, we have $(\tilde{k}, Ag_t - Ag') > 0 \, \forall \tilde{k} \in \bar{\tilde{U}}$. Hence, for $\tilde{k} \in \tilde{U}$,

$$\begin{aligned}\max_{f\in F} (\tilde{k}, Af - Ag_t) &= \max_{f\in F} (\tilde{k}, Af - Ag') - (\tilde{k}, Ag_t - Ag') \\ &< \max_{f\in F} (\tilde{k}, Af - Ag') \leqslant d_2(g').\end{aligned} \tag{2.7}$$

For $\tilde{k} \in \tilde{K}\setminus\tilde{U}$,

$$\begin{aligned}\max_{f\in F} (\tilde{k}, Af - Ag_t) &\leqslant \max_{f\in F} (\tilde{k}, Af - Ag') + p_2(Ag' - Ag_t) \\ &\leqslant d_2(g') - c + p_2(Ag' - Ag_t) < d_2(g')\end{aligned} \tag{2.8}$$

for $0 < t$ sufficiently small and continuity of A and p_2. Combining (2.5), (2.6), (2.7) and (2.10), we have $d_F(g_t) < d_F(g')$. This shows that g' is not a best approximation. The theorem is thus proved.

Note that in case A is a linear operator, then, obviously, A has the closed-sign property on G.

3. Fréchet—Differentiable Operator

In this section, we will consider the operator A possessing a Fréchet derivative. This derivative, in fact, is a linear operator from X into Y, for each fixed $g \in X$, denoted by A_g'. That is, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\| A(g + h) - Ag - A_g'h \| \leqslant \epsilon \| h \|$$

for all $h \in X$ with $\| h \| < \delta$. Therefore, for fixed $h(\neq 0) \in X$, setting

$$\delta(t) = (A(g + th) - Ag)/t - A_g'h \text{ for real } t,$$

we have $\delta(t) \in Y$ satisfying $\lim_{|t| \to 0} \| \delta(t) \| = 0$. Thus, we have the following theorem:

Theorem 3.1. *Suppose A has a Fréchet derivative at g' in convex set G and $d_1(g') = d_2(g')$. If g' is a best approximation, then for all $g \in G$*

$$\inf_{k \in B_{g'}, \tilde{k} \in \tilde{B}_{g'}} ((k, g - g'), (\tilde{k}A'_{g'}, g - g')) \leqslant 0. \tag{3.1}$$

Proof. Suppose there exists a $g_1 \in G$ such that

$$\inf_{k \in B_{g'}, \tilde{k} \in \tilde{B}_{g'}} ((k, g_1 - g'), (\tilde{k}A'_{g'}, g_1 - g')) > 0. \tag{3.2}$$

As $B_{g'}$ and $\tilde{B}_{g'}$ are $\sigma(X^*, X)$-compact and $\sigma(Y^*, Y)$-compact, respectively, there exist relatively open sets U, $\tilde{U}$ of K, $\tilde{K}$, containing $B_{g'}$, $\tilde{B}_{g'}$, respectively, such that,

$$\begin{aligned} \forall k \in U, \qquad & (k, g_1 - g') \geqslant a, \\ \forall \tilde{k} \in \tilde{U}, \qquad & (\tilde{k}A'_{g'}, g_1 - g') \geqslant a, \end{aligned} \tag{3.3}$$

for some $a > 0$.

Since K, $\tilde{K}$ are $\sigma(X^*, X)$-compact and $\sigma(Y^*, Y)$-compact, respectively, there exist real numbers c_1, $c_2 > 0$ such that

$$\max_{k \in K \setminus U} \max_{f \in F} (k, f - g') = d_1(g') - c_1,$$

$$\max_{\tilde{k} \in \tilde{K} \setminus \tilde{U}} \max_{f \in F} (\tilde{k}, Af - Ag') = d_2(g') - c_2.$$

Write $g_t = g' + t(g_1 - g')$, $0 < t < 1$ and

$$\max_{f \in F} (k, f - g_t) = \max_{f \in F} (k, f - g') - t(k, g_1 - g'),$$

so for $k \in U$, $0 < t < 1$

$$\max_{f \in F} (k, f - g_t) < \max_{f \in F} (k, f - g') \leqslant d_1(g') \tag{3.4}$$

for $k \in K/U$,

$$\max_{f \in F} (k, f - g_t) \leqslant d_1(g') - c_1 + tp_1(g_1 - g') < d_1(g') \tag{3.5}$$

for $0 < t$ sufficiently small.

As A is Fréchet-differentiable at g', we have

$$Ag_t - Ag' = t[\delta(t) + A'_{g'}(g_1 - g')], \qquad \lim_{t \to 0} \| \delta(t) \| = 0.$$

From inequality (3.3), for $\tilde{k} \in \tilde{U}$,

$$\begin{aligned}(\tilde{k}, A'_{g'}(g_1 - g') + \delta(t)) &= (\tilde{k}A'_{g'}, g_1 - g') + (\tilde{k}, \delta(t)) \\ &\geqslant a - \| \tilde{k} \| \cdot \| \delta(t) \| > 0,\end{aligned}$$

for $0 < t$ sufficiently small, because $\tilde{K}$ is norm bounded and $\| \delta(t) \| \to 0$, as $t \to 0$. So for small $t > 0$, we have

$$(\tilde{k}, Ag_t - Ag') > 0 \; \forall \tilde{k} \in \tilde{U}. \tag{3.6}$$

Write

$$\max_{f \in F} (\tilde{k}, Af - Ag_t) = \max_{f \in F} (\tilde{k}, Af - Ag') - (\tilde{k}, Ag_t - Ag').$$

Hence, for small $t > 0$, $\forall \tilde{k} \in \tilde{U}$,

$$\max_{f \in F} (\tilde{k}, Af - Ag_t) < \max_{f \in F} (\tilde{k}, Af - Ag') \leqslant d_2(g') \tag{3.7}$$

$\forall \tilde{k} \in \tilde{K}/\tilde{U}$,

$$\max_{f \in F} (\tilde{k}, Af - Ag_t) \leqslant d_2(g') - c_2 + p_2(Ag_t - Ag') < d_2(g'), \quad (3.8)$$

for small $t > 0$ and continuity of A and p_2.

Combining (3.4), (3.5), (3.7), and (3.8), we have $d_F(g_t) < d_F(g')$. Thus, g' is not a best approximation, which proves the theorem.

THEOREM 3.2. *Suppose A has a Fréchet derivative at g' in convex set G and $d_1(g') = d_2(g')$. Suppose further that A satisfies either of the following conditions:*

(i) *A has the closed-sign property at g' and there exists $g \in G$ such that $(\tilde{k}A'_{g'}, g - g') > 0$ for all $\tilde{k} \in \tilde{B}_{g'}$.*

(ii) *$d_2(g)$ is convex on G.*

Then, if

$$\inf_{k \in B_{g'}, \tilde{k} \in \tilde{B}_{g'}} ((k, g - g'), (\tilde{k}A'_{g'}, g - g')) \leqslant 0 \qquad \forall g \in G,$$

g' is a best approximation.

Proof. Suppose g' is not a best approximation and A satisfies condition (i). Then, by virtue of Theorem 2.1, there exists $g_1 \in G$ such that

$$\inf_{k \in B_{g'}, \tilde{k} \in \tilde{B}_{g'}} ((k, g_1 - g'), (\tilde{k}, Ag_1 - Ag')) > 0.$$

For $0 < t < 1$, we have

$$A(g' + t(g_1 - g')) - Ag' = tA'_{g'}(g_1 - g') + t\,\delta(t).$$

As A is Fréchet-differentiable at g', $\lim_{t \to 0} \|\delta(t)\| = 0$. Moreover, A has the closed-sign property at g', therefore, there exists $1 > \delta_0 > 0$ such that, for $t \in (0, \delta_0)$

$$\operatorname{sgn}(\tilde{k}, Ag_1 - Ag') = \operatorname{sgn}(\tilde{k}, Ag_t - Ag') \text{ for all } \tilde{k} \in \tilde{B}_{g'}$$

where $g_t = g' + t(g_1 - g')$.

Thus, $(\tilde{k}, Ag_t - Ag') > 0$ for all $\tilde{k} \in \tilde{B}_{g'}$ $t \in (0, \delta_0]$. Therefore we may conclude that $(\tilde{k}A'_{g'}, g_1 - g') \geqslant 0$ for all $\tilde{k} \in \tilde{B}_{g'}$. If $(\tilde{k}A'_{g'}, g_1 - g') > 0$ for all $\tilde{k} \in \tilde{B}_{g'}$, then we have

$$\inf_{k \in B_{g'}, \tilde{k} \in \tilde{B}_{g'}} ((k, g_1 - g'), (\tilde{k}A'_{g'}, g_1 - g')) > 0, \tag{3.9}$$

which proves the sufficiency. Otherwise, if $(\tilde{k}A'_{g'}, g_1 - g') = 0$. for some $\tilde{k} \in \tilde{B}_{g'}$, then, by the hypothesis, there is a $g_2 \in G$ such that

$$(\tilde{k}A'_{g'}, g_2 - g') > 0 \qquad \text{for all} \quad \tilde{k} \in \tilde{B}_{g'}.$$

Since $B_{g'}$ is $\sigma(X^*, X)$-compact, there exists real number $c > 0$ such that

$$c = \min_{k \in B_{g'}} (k, g_1 - g').$$

Therefore, for any $0 < \lambda < \delta_0$ sufficiently small, we have

$$(k, \lambda g_2 + (1 - \lambda) g_1 - g') = (k, g_1 - g') + \lambda(k, g_2 - g_1) > 0 \qquad \text{for all} \quad k \in B_{g'},$$

and

$$\begin{aligned}(\tilde{k}A'_{g'}, \lambda g_2 + (1 - \lambda)\, g_1 - g') &= \lambda(\tilde{k}A'_{g'}, g_2 - g') + (1 - \lambda)(\tilde{k}A'_{g'}, g_1 - g') \\ &> 0 \qquad \text{for all} \quad \tilde{k} \in \tilde{B}_{g'}.\end{aligned}$$

As G is convex, $\lambda g_2 + (1 - \lambda)\, g_1 \in G$. Therefore, this again shows that, if g' is not a best approximation, then there exists a $g \in G$ satisfying inequality (3.9).

On the other hand, suppose g' is not a best approximation and A satisfies condition (ii). Then there exists $g_1 \in G$ such that $d_F(g_1) < d_F(g')$. Therefore, for each $k \in B_{g'}$, there exists $f_1 \in F$ (which depends on k) such that

$$(k, f_1 - g') = d_F(g')$$

and

$$\begin{aligned}(k, g_1 - g') &= (k, f_1 - g') - (k, f_1 - g_1) \\ &\geqslant d_1(g') - \max_{f \in F} (k, f - g_1) \\ &\geqslant d_1(g') - d_F(g_1) \\ &> 0.\end{aligned}$$

Similarly, for each $\tilde{k} \in \tilde{B}_{g'}$, there exists $f_2 \in F$ (which depends on $\tilde{k}$) such that $(\tilde{k}, Af_2 - Ag') = d_F(g')$. Then, if $t > 0$ and

$$g_t = g' + t(g_1 - g'), \qquad \delta(t) = (1/t)\{Ag_t - Ag'\} - A'_{g'}(g_1 - g'),$$

we have

$$\begin{aligned}(\tilde{k}, A'_{g'}(g_1 - g') + \delta(t)) &= (1/t)(\tilde{k}, Ag_t - Ag') \\ &= (1/t)\{(\tilde{k}, Af_2 - Ag') - (\tilde{k}, Af_2 - Ag_t)\} \\ &\geqslant (1/t)\{d_2(g') - \max_{f \in F} (\tilde{k}, Af - Ag_t)\} \\ &\geqslant (1/t)\{d_2(g') - d_2(g_t)\}.\end{aligned}$$

As $d_2(g)$ is convex on G, we have

$$\begin{aligned}(\tilde{k}, A'_{g'}(g_1 - g') + \delta(t)) &\geqslant (1/t)\{d_2(g') - t\, d_2(g_1) - (1 - t)\, d_2(g')\} \\ &= d_2(g') - d_2(g_1) \\ &> 0.\end{aligned}$$

As A is Fréchet-differentiable at g', we have $\lim_{|t| \to 0} \| \delta(t) \| = 0$. Consequently,

$$(\tilde{k}A'_{g'}, g_1 - g') > 0.$$

This again shows that if g' is not a best approximation and A satisfies condition (ii), there exists $g_1 \in G$ such that

$$\inf_{k \in B_{g'}, \tilde{k} \in \tilde{B}_{g'}} ((k, g_1 - g'), (\tilde{k}A'_{g'}, g_1 - g')) > 0,$$

which proves the theorem.

If G is a subspace, we have the following:

THEOREM 3.3. *Let G be a subspace of X, A Fréchet–differentiable at $g' \in G$. Then* 0 *belongs to the $\sigma(G^*, G)$-closure of the convex hull of*

$$B_{g'}|_G \cup D_{g'}|_G$$

if and only if, for all $g \in G$

$$\inf_{k \in B_{g'}, \tilde{k} \in \tilde{B}_{g'}} ((k, g), (\tilde{k}A'_{g'}, g)) \leqslant 0,$$

where $D_{g'} = \{\tilde{k}A'_{g'} : \tilde{k} \in \tilde{B}_{g'}\}$ and $B_{g'}|_G$, $D_{g'}|_G$ are the restrictions of the functionals of $B_{g'}$, $D_{g'}$ to G, respectively.

Proof. Suppose 0 is not in the $\sigma(G^*, G)$-closure of the convex hull of $B_{g'}|_G \cup D_{g'}|_G$, then, by a known result in [4, Theorem 10, p. 417] and a known result that the dual of G^* under the $\sigma(G^*, G)$-topology is G, there exists an element $g_1 \in G$ such that

$$\inf_{k \in B_{g'}, \tilde{k} \in \tilde{B}_g} ((k, g_1), (\tilde{k}A'_{g'}, g_1)) > 0.$$

This shows that $\inf_{k \in B_{g'}, \tilde{k} \in \tilde{B}_g} ((k, g_1), (\tilde{k}A'_{g'}, g_1)) \leqslant 0$ is a sufficient condition for 0 belongs to the $\sigma(G^*, G)$-closure of the convex hull of $B_{g'} |_G \cup D_{g'} |_G$.

Conversely, suppose that there exists $g_0 \in G$ such that

$$(\gamma, g_0) > 0 \; \forall \, \gamma \in B_{g'} |_G \cup D_{g'} |_G .$$

As $B_{g'}$, $\tilde{B}_{g'}$ are $\sigma(X^*, X)$-compact, there is a real $c > 0$ such that

$$(\gamma, g_0) \geqslant c \; \forall \, \gamma \in B_{g'} \cup D_{g'} .$$

Since any $\varphi \in \overline{co}(B_{g'} |_G \cup D_{g'} |_G)$ can be written as

$$\varphi = \lim_{\alpha} \Big(\sum_i \lambda_i^{\alpha} \gamma_i \Big)$$

for some real $\lambda_i^{\alpha} > 0$, $\gamma_i \in B_{g'} |_G \cup D_{g'} |_G$ and $\sum_i \lambda_i^{\alpha} = 1$, we have

$$\varphi(g_0) = \lim_{\alpha} \Big(\sum_i \lambda_i^{\alpha} \gamma_i(g_0) \Big) \geqslant c > 0.$$

As φ is arbitrary, we may conclude that

$$\varphi(g_0) \geqslant c > 0 \qquad \forall \varphi \in \overline{co}(B_{g'} |_G \cup D_{g'} |_G).$$

Thus, 0 cannot belong to the $\sigma(G^*, G)$-closure of the convex hull of $B_{g'} |_G \cup D_{g'} |_G$, which completes the proof.

THEOREM 3.4. *Let G be a subspace of X, A Fréchet-differentiable at $g' \in G$, and $d_1(g') = d_2(g')$.*

(i) *If g' is a best approximation, then 0 belongs to the $\sigma(G^*, G)$-closure of the convex hull of $B_{g'} |_G \cup D_{g'} |_G$.*

(ii) *Suppose that either A has the closed sign property at g' and there exists $g \in G$ such that $(\tilde{k}A'_{g'}, g) > 0 \; \forall \, \tilde{k} \in \tilde{B}_{g'}$ or $d_2(g)$ is convex on G. Then, if 0 belongs to the $\sigma(G^*, G)$-closure of the convex hull of $B_{g'} |_G \cup D_{g'} |_G$, g' is a best approximation.*

In fact, the theorem follows immediately from Theorems 3.1–3.3.

4. Finite Dimensional Subspace

We will now consider the case where G is an n-dimensional subspace. Let us first define a mapping $\phi : X^* \to R^n$ as

$$\phi(k) = ((k, g_1),..., (k, g_n)),$$

where $g_1 ,..., g_n$ is a basis for G. Obviously, ϕ is a continuous linear map. For convenience, we write $B_{g'}$, $D_{g'}$ to mean that $B_{g'}|_G$, $D_{g'}|_G$ throughout this section. Obviously, $B_{g'}$ and $D_{g'}$ are $\sigma(G^*, G)$-compact, $\phi(B_{g'})$ and $\phi(D_{g'})$ are compact. Moreover, since $\overline{co}(B_{g'} \cup D_{g'})$ is $\sigma(G^*, G)$-compact, it is easy to check that

$$\phi(\overline{co}(B_{g'} \cup D_{g'})) = co(\phi(B_{g'} \cup D_{g'})),$$

where the closure is taken under $\sigma(G^*, G)$-topology.

This enables us to deduce the following results:

Theorem 4.1. *Let g' be an element of an n-dimensional subspace G of X such that $d_1(g') = d_2(g')$ and F a compact subset of X. Assume that the operator A is Fréchet-differentiable. If g' is a best approximation, then there exist s functionals $k_i \in B_{g'}$ $(i = 1, 2,..., s)$, t functionals $\tilde{k}_i \in \tilde{B}_{g'}$ $(i = 1, 2,..., t)$ and $s + t$ positive real numbers $a_1 ,..., a_{s+t}$ such that $s + t \leqslant n + 1$, $\sum_{i=1}^{s+t} a_i = 1$ and*

$$\sum_{i=1}^{s} a_i(k_i , g) + \sum_{i=1}^{t} a_{s+i}(\tilde{k}_i A'_{g'} , g) = 0 \qquad \text{for all} \quad g \in G. \tag{4.1}$$

Furthermore, suppose that either A has the closed-sign property at g' and there exist $g \in G$ such that $(\tilde{k}A'_{g'} , g) > 0$ for all $\tilde{k} \in \tilde{B}_{g'}$ or $d_2(g)$ is convex on G, then the condition (4.1) *implies that g' is a best approximation.*

Proof. We define a continuous linear mapping $\phi : X^* \to R^n$ as above. Moreover, we have pointed out that $\phi(\overline{co}(B_{g'} \cup D_{g'})) = co(\phi(B_{g'} \cup D_{g'}))$ where the closure is taken under $\sigma(G^*, G)$-topology. Since G is of finite dimension, we have the following trivial equivalence:

$$0 \in \overline{co}(B_{g'} \cup D_{g'}) = co(B_{g'} \cup D_{g'})$$

if and only if $(0,..., 0) \in co[\phi(B_{g'} \cup D_{g'})]$. By virtue of Caratheodory's representation theorem [2, p. 17], $(0,..., 0) \in co[\phi(B_{g'} \cup D_{g'})]$ implies that there exist $k_i \in B_{g'}$ $(i = 1, 2,..., s)$, $\tilde{k}_i A'_{g'} \in D_{g'}$ $(i = 1, 2,..., t)$ and positive numbers $a_i (i = 1, 2,..., s + t)$ such that $s + t \leqslant n + 1$, $\sum_{i=1}^{s+t} a_i = 1$ and

$$\sum_{i=1}^{s} a_i((k_i , g_1),..., (k_i , g_n)) + \sum_{i=1}^{t} a_{s+i}((\tilde{k}_i A'_{g'} , g_1),..., (\tilde{k}_i A'_{g'} , g_n)) = 0.$$

By scalar multiplication by any $c = (c_1, \ldots, c_n)$, we get

$$\sum_{i=1}^{s} a_i \left(k_i, \sum_{j=1}^{n} c_j g_j\right) + \sum_{i=1}^{t} a_{s+i} \left(\tilde{k}_i A'_{g'}, \sum_{j=1}^{n} c_j g_j\right) = 0$$

that is

$$\sum_{i=1}^{s} a_i(k_i, g) + \sum_{i=1}^{t} a_{s+i}(\tilde{k}_i A'_{g'}, g) = 0 \qquad \forall g \in G.$$

Thus, by virtue of Theorems 3.3 and 3.4, the theorem follows immediately.

COROLLARY 4.1. *Let G be an n-dimensional subspace and A a linear operator. Then, $g' \in G$ with $d_1(g') = d_2(g')$ is a best approximation if and only if there exist $k_i \in B_{g'}$ $(i = 1, 2, \ldots, s)$, $\tilde{k}_i \in \tilde{B}_{g'}$, $(i = 1, 2, \ldots, t)$ and $s + t$ positive numbers a_i $(i = 1, \ldots, s + t)$ such that $s + t \leqslant n + 1$, $\sum_{i=1}^{s+t} a_i = 1$ and*

$$\sum_{i=1}^{s} a_i(k_i, g) + \sum_{i=1}^{t} a_{s+i}(\tilde{k}_i A, g) = 0 \qquad \textit{for all} \quad g \in G.$$

We recall that a nonvoid subset Q of a compact set Z is said to be an extremal subset of Z if a proper convex combination $\lambda x_1 + (1 - \lambda) x_2$, $0 < \lambda < 1$, of two points $x_1, x_2 \in Z$ is in Q only if both x_1 and x_2 are in Q. An extremal subset consisting of exactly one point is called an extreme point.

THEOREM 4.2. *Let g' be an element of an n-dimensional subspace G of X such that $d_1(g') = d_2(g')$. Assume that the operator A possesses a Fréchet-derivative on G. If g' is a best approximation, then there exist s extreme functionals $k_i \in K$ $(i = 1, \ldots, s)$, t extreme functionals $\tilde{k}_i \in \tilde{K}$ $(i = 1, 2, \ldots, t)$, $s + t$ functions $f_1, \ldots, f_{s+t} \in F$ (not necessarily distinct) and $s + t$ positive numbers $a_1, \ldots, a_{s+t}$ such that $s + t \leqslant n + 1$, $\sum_{i=1}^{s+t} a_i = 1$,*

$$(k_i, f_i - g') = d_F(g') \qquad i = 1, 2, \ldots, s;$$
$$(\tilde{k}_i, Af_{s+i} - Ag') = d_F(g') \qquad i = 1, 2, \ldots, t;$$

and

$$\sum_{i=1}^{s} a_i(k_i, g) + \sum_{i=1}^{t} a_{s+i}(\tilde{k}_i A'_{g'}, g) = 0 \qquad \textit{for all} \quad g \in G. \tag{4.2}$$

Furthermore, suppose that either A has the closed-sign property at g' and there exist $g \in G$ such that $(\tilde{k} A'_{g'}, g) > 0$ for all $\tilde{k} \in \tilde{B}_{g'}$ or $d_2(g)$ is convex on G. Then the condition (4.2) implies that g' is a best approximation.

Proof. By virtue of Theorem 3.4 we know that if g' is a best approximation then 0 belongs to the $\sigma(G^*, G)$-closure of the convex hull of $B_{g'} \cup D_{g'}$. Furthermore, by the assumption on A, 0 belongs to the $\sigma(G^*, G)$-closure of the convex hull of $B_{g'} \cup D_{g'}$ also implies g' is a best approximation. Hence, if we define a continuous linear map $\phi : X^* \to R^n$ as in the beginning of this section, we know that $\phi(\overline{co}(B_{g'} \cup D_{g'})) = co(\phi(B_{g'} \cup D_{g'}))$, where the closure is taken under $\sigma(G^*, G)$-topology. It follows at once that 0 is in

$$co(\phi(B_{g'} \cup D_{g'})).$$

Since $\phi(B_{g'} \cup D_{g'})$ is a compact subset of R^n, by combining Caratheodory's Theorem and Krein–Milman's Theorem [4, p. 440], we know that

$$0 \in co(\phi(B_{g'} \cup D_{g'}))$$

if and only if there exist q extreme points $w_1, ..., w_q \in \phi(B_{g'} \cup D_{g'})$ and $a_1, ..., a_q > 0$ such that $\sum_{i=1}^{q} a_i = 1$, $q \leqslant n + 1$ and

$$\sum_{i=1}^{q} a_i w_i = 0.$$

By a known result [1, p. 159], if T is a continuous linear map from X into Y and Q is a compact subset of X, then, for every extreme point y in $T(Q)$ there exists at least one extreme point w in Q such that $T(w) = y$, therefore, there exist q extreme functionals l_i in $B_{g'} \cup D_{g'}$ such that $y_i = T(l_i)$.

We denote by k_i the functionals l_i which are extreme points in $B_{g'}$ and by $\tilde{k}_i A'_{g'}$ the others. Hence, we get, $\sum_{i=1}^{s} a_i(k_i, g) + \sum_{i=1}^{t} a_{s+i}(\tilde{k}_i A'_{g'}, g) = 0$ for all $g \in G$ and $s + t = q$. Moreover, $B_{g'}$ is an extremal subset of K, so the extreme points of $B_{g'}$ are the extreme points of K. On the other hand, it is easy to check that $\tilde{k}_i A'_{g'}$ is an extreme point of $D_{g'}$ only if $\tilde{k}_i$ is an extreme point of $\tilde{B}_{g'}$. In the same way, since $\tilde{B}_{g'}$ is an extremal subset of $\tilde{K}$, the extreme points of $\tilde{B}_{g'}$ are again the extreme points of $\tilde{K}$. Consequently, k_i are the extreme functionals of K and $\tilde{k}_i$ are the extreme functionals of $\tilde{K}$. This concludes the proof of the theorem.

Remark. We note that if the problem under consideration is to seek an element in G which minimizes $\max(\max_{f \in F_1} p_1(f - g), \max_{h \in F_2} p_2(h - Ag))$, where F_1 and F_2 are given compact subsets of X and Y, respectively, then all the preceding results which have been discussed follow at once without any further assumption. Furthermore, we may generalise the previous theory to the following general case: Let $X, Y_1, Y_2, ..., Y_n$ be $n + 1$ $(n > 1)$ given normed linear spaces, A_i $(i = 1, ..., n)$ continuous maps from X into Y_i $(i = 1, 2, ..., n)$, and $p_i(\cdot)(i = 0, 1, ..., n)$ be given continuous seminorms on

$Y_i (i = 0, 1,..., n)$ where $Y_0 = X$. The question is to determine an element g' of $G \subset X$ which minimizes

$$\max(\max_{f \in F} p_0(f - g), \max_{f \in F} p_1(A_1 f - A_1 g),..., \max_{f \in F} p_n(A_n f - A_n g)).$$

All the previous results can be applied to this general case, the proofs are similar, therefore, we do not go into details.

Acknowledgment

I am indebted to the referee for his valuable suggestions.

References

1. N. Bourbaki, "Éléments de Mathématiques," Espaces Vectoriels Topologiques Chap. 1 and 2, fascicule XV, Hermann, Paris, 1966.
2. E. W. Cheney, "Introduction to Approximation Theory," McGraw-Hill, New York, 1966.
3. J. B. Diaz and H. W. McLaughlin, Simultaneous approximation of a set of bounded real function, *Math. Comp.* **23** (1969), 583–594.
4. N. Dunford and J. T. Schwartz, "Linear Operator," Part I, Interscience, New York, 1958.
5. C. B. Dunham, Simultaneous Chebyshev approximation of functions on an interval, *Proc. Amer. Math. Soc.* **18** (1967), 472–477.
6. C. B. Dunham, Characterizability and uniqueness in real Chebyshev approximation, *J. Approximation Theory* **2** (1969), 374–383.
7. E. Hewitt and K. Stromberg, "Real and Abstract Analysis," Springer, Berlin, 1965.
8. P. J. Laurent and Pham-Dinh-Tuan, Global approximation of a compact set by elements of a convex set in a normed space, *Numer. Math.* **15** (1970), 137–150.
9. A. Meir and A. Sharma, Simultaneous approximation of a function and its derivatives, *SIAM J. Numer. Anal.* **3** (1966), 553–563.
10. D. G. Moursund, Chebyshev approximation of a function and its derivative, *Math. Comp.* **18** (1964), 382–389.
11. D. G. Moursund and A. H. Stroud, The best approximation to a function and its derivatives on $n + 2$ points, *SIAM J. Numer. Anal.* **2** (1965), 15–23.
12. I. Singer, "Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces," Springer, New York, 1970.